LINEARIZATION OPERATORS,THE DIRAC EQUATION AND RIEMANN'S HYPOTHESIS

L.E. ROBERTS

Department of Physics, Lincoln University – Lincoln University, PA 19352

ABSTRACT - A new technique is introduced here and used to formulate the Dirac Equation,from Einstein's extreme relativistic equation,(to do this we must additionally invoke heuristic quantization), and prove Riemann's Hypothesis via linearization operators of the 1^{st} and 2^{nd} kinds, respectively, which are also introduced here.

Keywords:

Other topics in mathematical methods in physics.
Special functions.
Foundations of Relativistic Quantum Theory.

1. – Introduction

This work has been motivated by the works of three icons of the mathematical and physical sciences. Albert Einstein, G.F.B. Riemann and P.A.M. Dirac are unquestionably three of the greatest luminaries that have taken one step beyond,synthesizing groundbreaking theoretical ideas, having far-reaching consequences into the present day and, more than likely, beyond.

The works of Albert Einstein,G.f.B. Riemann and P.A.M. Dirac are fused together into the fabric of reality that every present day scientist,in general, and every current theoretical physicist in particular,(especially those working on the frontiers of Elementary Particle Physics,Cosmology and many other disciplines), is trying to unravel.

Albert Einstein's work in Special Relativity, specifically his Extreme Relativistic Equation, led to attempts to develop a self-consistent relativistic quantum theory leading ultimately, through the work of O. Klein, W. Gordon and P.A.M. Dirac ,to the bridging of the chasm between a low energy,large wavelength ,(non-relativistic), Theory, and the high energy extremely small wavelength theory which ultimately would describe elementary particle production and annihilation in the sub-microscopic realm[1,2] .

One would like for the high energy theory to arise from first principles. This is problematic in that at most a plausibility argument can be given to justify the non-relativistic theory [3]. In spite of this prodigious constraint, O.Klein and W.Gordon were able to postulate the Klein-Gordon Equation,(which describes particles obeying the Bose-Einstein Statistics ie.Bosons),while P.A.M. Dirac,once again, was able to postulate a second relativistic quantum mechanical equation known as the Dirac Equation,(which describes elementary particles obeying the Fermi-Dirac Statistics ie.Fermions).These two equations give rise to Quantum Field Theory when they are second quantized. However Dirac's procedure to develop the equation named after him was somewhat artificial.

In this paper a new technique is developed and introduced which,firstly, while not eliminating the necessity for using the plausibility argument that plagues all quantum theories, does allow one to arrive at the Dirac Equation in a more aesthetically pleasing way in that a square root no longer enters into the considerations necessary to develop this formalism.

G.F.B.Riemann's Zeta function has applications to many branches of mathematical analysis and Theoretical Physics. Classical Mechanics,Statistical Mechanics,Quantum Chaos,Bose-Einstein Condensation, Subatomic Physics,String Theory,M-Theory as well as a plethora of other fields in theoretical physics have been analyzed by systematic applications Of the Riemann Zeta Function . In fact it would be difficult to imagine that Riemann could have envisioned the richness of the applications that posterity has shown his zeta function to have. However there is still at least one enigma involving the Riemann Zeta Function; the zeroes of the real part of it.

The second achievement of this work is a new approach which will be applied to The Riemann Zeta Function to find the real part of its zeroes, thus proving Riemann's Hypothesis.

To achieve the above,very ambitious, program we introduce two new operators in section **2**.

2. Representations Of The Linearization Operators That Are Necessary For This Analysis

In this work we will discuss the two different Linearization operators that must be introduced. The first occurs when relativistic covariance is required. An equation described by the following functional relationship is encountered:

$$f\,(A^{(0)}, A^{(k)}, A^{(2\prime)}) = 0 \quad . \qquad (1)$$

To linearize equation (1) The Linearization operator has the following representation:

$$L^{(1)}\left(\mathrm{A}^{(i)}\right)^2 \equiv \alpha_{1i}\, A^{(i)} log_{A^{(i)}}\left(A^{(i)}\right)^2 \hat{A}^{(i)} \qquad (2)$$

where the α_{1k} are expansion parameters required to maintain relativistic covariance as well as the internal consistency of equation(1).

Also note that if k=2′ we make a change in notation $\alpha_{12'} \equiv \alpha_{2'}$ so as to avoid confusion.

The second form of the Linearization operator occurs when relativistic covariance is not required (i.e. four-vectors are not present) In this case one encounters *completely factorized* terms to be linearized of form

A= ${A_1}^{\alpha_1(term_1)} A_2^{\alpha_2(term_2)}$ *where* the representation of the Linearization operator is

$${}^{(2)}_{(S)}L[\ \mathrm{A}\ ] \equiv \ln\left[\prod_{j=1}^{2} expL_j\ A_j^{\alpha_j term_j}\right] \qquad (3)$$

such that

$${}^{(2)}_{(S)}L[\mathrm{A}] = L_1 A_1^{\alpha_1 term_1} + L_2 A_2^{\alpha_2 term_2} \qquad (4)$$

where we are operating in the base S . We will define the operators L_1 and L_2

after further decomposing $A_1^{\alpha_1 term_1}$:

$$A_1^{\alpha_1 term_1} = A_{11}^{\alpha_{11} term_{11}}\ A_{12}^{\alpha_{12} term_{12}} \qquad (5)$$

where A_{11} must be integral.

The L_1 operator has the following definition:

$$L_1[A_1^{\alpha_1 term_1}] \equiv \Pi_{j=1}^{2}[(1-\delta_{\alpha_{1j,i}})\ L_{1j}\ A_{1j}^{\alpha_{1j} term_{1j}}\ ] \qquad (6a)$$

with the L_2 operator being similarly defined

$$L_2[A_2^{\alpha_2 term_2}] \equiv A_2(log_{A_2} A_2^{\alpha_2 term_2}) \quad (1-\delta_{\alpha_{2,i}}) \qquad . \qquad (6b)$$

The L_{11} and L_{12} operators must next be specified:

$$L_{11}[A_{11}^{\alpha_{11} term_{11}}] \equiv A_{11} log_{A_{11}}[A_{11}^{\alpha_{11} term_{11}}] \qquad (7a)$$

and

$$L_{12}[A_{12}^{\alpha_{12} term_{12}}] \equiv A_{12}(log_{A_{12}}[A_{12}^{\alpha_{12} term_{12}}])\delta_{A_{12},S} \qquad (7b)$$

where L_{12} is operating in the base S.

For the Proof of Riemann's Hypothesis the second representation therefore is appropriate.

3. –Einstein's Extreme Relativistic Equation And The Search For A Consistent Relativistic Quantum Theory.

Consider Einstein's Extreme Relativistic Equation:

$$E^2 = p^2c^2 + m_0^2c^4 \qquad . \qquad (8)$$

Firstly, let's work in "natural units" ħ=c=1, so that equation (8) becomes

$$E^2 = p^2 + m_0^2 \qquad . \qquad (9)$$

In four-vector notation

$P_k = (p^{(0)}, p^{(k)})$ (10)

where $p^{(0)} \equiv \mathrm{E}$, We also wish to make the definition $p^{(2\prime)} \equiv m_0$.

In equation (10) following Dirac,we make the following definition:

$$P_k P^k = \left(p^{(0)}\right)^2 - (p_k)^2 \equiv \left(p^{(2\prime)}\right)^2 \qquad (11)$$

which reproduces equation (9) and we next apply $L^{(1)}$:

$$L^{(1)}\left(p^{(0)}\right)^2 = L^{(1)}(p_k)^2 + L^{(1)}\left(p^{(2\prime)}\right)^2 \qquad (12)$$

such that

$$\alpha_{10} p^{(0)} log_{p^{(0)}}\left(p^{(0)}\right)^2 \hat{p}^{(0)} = \alpha_{1k} p^{(k)} log_{p_l}(p_l)^2 \hat{p}^{(k)} + \alpha_{2\prime} p^{(2\prime)} log_{p^{(2\prime)}}\left(p^{(2\prime)}\right)^2 \hat{p}^{(2\prime)} . \quad (13)$$

Now in “natural units” $[p^{(0)}] = [p^{(k)}] = [p^{(2\prime)}]$ where in this expression [ ] ≡ “the dimensions of”

such that the base is the same throughout equation (13) meaning that $\hat{p}^0 = \hat{p}^{(k)} = \hat{p}^{(2\prime)}$ = (where the tilde, ie “ ˆ “, means unit vector).

Hence we see that

$$2\,\alpha_{10}\mathrm{E} = 2\alpha_{1k} p^k + 2\alpha_{2\prime} m_0 \quad . \qquad (14)$$

Note that the total mechanical energy is already relativistically covariant so that

$\alpha_{10} = 1$ while the p^k are not so that α_{1k} can’t be unity and must be nonzero. Additionally while m_0 is relativistically covariant
$\alpha_{2\prime}, also\ can't\ be\ unity\ or\ one\ for\ consistency\ between\ equations$ (9)$thru$ (14).Therefore we make a change of notation; $\alpha_{2\prime} \equiv \beta$ so that equation (14) becomes

$$\mathrm{E} = \alpha_{1k} p^k + \beta m_0 \quad . \qquad (15)$$

Operate to the right on equation (15) with the state function Ψ to get:

$$\mathrm{E}\Psi = \alpha_{1k} p^k \Psi + \beta m_0 \Psi \quad . \qquad (16)$$

Next invoke heuristic quantization; $\mathrm{E} \rightarrow E^{op} = \mathrm{i}\frac{\partial}{\partial t}$ and $\mathrm{p} \rightarrow p^{op} = \mathrm{i}\sum_{k=1}^{3}\frac{\partial}{\partial x^k}$ where x^1 = x , x^2 = y and x^3 = z thus yielding

$$\mathrm{i}\frac{\partial \Psi}{\partial t} = \mathrm{i}\left(\alpha_{11}\frac{\partial \Psi}{\partial x} + \alpha_{12}\frac{\partial \Psi}{\partial y} + \alpha_{13}\frac{\partial \Psi}{\partial z}\right) + \beta m_0 \Psi \qquad (17)$$

which is the celebrated **Dirac equation** in “natural units”. In equation (17) the α_{1k} are related to the Pauli Spin Matrices and $\beta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

4. Riemann’s Zeta Function And Riemann’s Hypothesis

The Riemann Zeta Function [4-7] is given by

$$\zeta(z) = \sum_{k=1}^{\infty}\frac{1}{k^z} \quad . \qquad (18)$$

Now the complex variable z = x + i y (19)
So that

$$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \ldots\ldots\ldots\ldots\ldots \qquad (20)$$

which, then can be written as

$$\zeta(z) = \sum_{k=1}^{\infty}\frac{1}{k^{x+iy}} \quad . \qquad (21)$$

At z = $z_0 = x_0 + \mathrm{i}\, y_0$ the Riemann Zeta Function vanishes:

$$\zeta(z_0) = \sum_{k=1}^{\infty}\frac{1}{k^{x_0+iy_0}} = 0. \qquad (22)$$

Equation (22) can be written as :

$$\frac{1}{1^{x_0+iy_0}}+\frac{1}{2^{x_0+iy_0}}+\frac{1}{3^{x_0+iy_0}}+\frac{1}{4^{x_0+iy_0}}+\ldots\ldots\ldots\ldots\ldots= 0. \quad (23)$$

Riemann's Hypothesis [4,5-11] is that except for trivial cases the real part of the zero of the Riemann Zeta Function is given by $x_0 = \frac{1}{2}$.

All terms in equation (23) are of two possible forms:

$$\left(\frac{1}{1^{x_0+iy_0}}+\frac{1}{(2m)^{x_0+\,iy_0}}\right) \qquad (24a)$$

And

$$\left(\frac{1}{(2n)^{x_0+\,iy_0}}\right) \qquad (24b)$$

where m and n are scalar functions and m= I/2 where Iϵ{Integers | 2≤I≤∞}. Set each of these terms (24a) and (24b) equal to constants $(k_{1m})^b$and $(\mathrm{k}_{2\mathrm{n}})^{\mathrm{d}} + \alpha\mathrm{e}^{\mathrm{c}}$ respectively where $\mathrm{k}_{1\mathrm{m}}$, $\mathrm{k}_{2\mathrm{n}}$ are real,nonzero constants as are b, d,α and c. Also k_{1m}≠k_{2n}≠b≠d≠α≠c.

Hence we see that

$$\left(\frac{1}{1^{x_0+iy_0}}+\frac{1}{(2m)^{x_0+\,iy_0}}\right) \quad = \quad (k_{1m})^b \qquad (25a)$$

and

$$\left(\frac{1}{(2n)^{x_0+\,iy_0}}\right) \quad = \quad (\mathrm{k}_{2\mathrm{n}})^{\mathrm{d}} + \alpha\mathrm{e}^{\mathrm{c}} \qquad (25b)$$

It will be shown that there are two possible choices for n : case a) n = 2m and case b) n= $n^{'}\neq$ 2m.

5. The Unravelling Of The Riemann's Hypothesis Enigma.

The new approach is to linearize equations (25a) and (25b) and furthermore we have the freedom to choose

$${}^{(2)}_{(S)}L\ [\ (k_{1m})^{b}\ \ ] \ = -\ {}^{(2)}_{(S)}L[\ [(k_{2n})^{d} + \alpha e^{c}\ ] \qquad (26)$$

where ${}^{(2)}_{(s)}L$ Is the Linearization operator of the 2^{nd} kind.

We specialize to the case S=2m , thus yielding

$${}^{(2)}_{(2m)}L\ [\frac{1}{1^{x_0+iy_0}} + \frac{1}{(2m)^{x_0+iy_0}}] \ = \ -\ {}^{(2)}_{(2m)}L[\frac{1}{(2n)^{x_0+iy_0}}] \qquad (27)$$

Recall that we will show that there are two choices for n; either a)n=2m or b) n $=n' \neq$2m.

In case a) n=2m:

Equation (27) becomes

$${}^{(2)}_{(2m)}L[\ \frac{1}{1^{x_0+iy_0}} + \frac{1}{(2m)^{x_0+iy_0}}] \ = -\ {}^{(2)}_{(2m)}L\ [\frac{1}{(4m)^{x_0+iy_0}}]\ . \qquad (28)$$

In case b) n=n´≠ 2m:

Equation (27) becomes

$${}^{(2)}_{(2m)}L\left[\frac{1}{1^{x_0+iy_0}} + \frac{1}{(2\mathrm{m})^{x_0+iy_0}}\right] = -\ {}^{(2)}_{(2m)}L[\frac{1}{(2n\prime)^{x_0+iy_0}}] \quad . \qquad (29)$$

Case a) For n=2m:

$$ {}_{(2m)}^{(2)}L[\frac{1}{1^{x_0+iy_0}} + \frac{1}{(2m)^{x_0+iy_0}}] = - {}_{(2m)}^{(2)}L[\frac{1}{(4m)^{x_0+iy_0}}] \quad . \qquad (30) $$

We will evaluate equation (30) term by term.

$$ {}_{(2m)}^{(2)}L[\frac{1}{1^{x_0+iy_0}} + \frac{1}{(2m)^{x_0+iy_0}}] = {}_{(2m)}^{(2)}L\ [\frac{1}{1^{x_0+iy_0}}] + {}_{(2m)}^{(2)}L[\frac{1}{(2m)^{x_0+iy_0}}] $$

$$ = {}_{(2m)}^{(2)}L[1^{-x_0}\ 1^{-iy_0}] + {}_{(2m)}^{(2)}L[(2m)^{-x_0}\ (2m)^{-iy_0}] \qquad (31) $$

Where in the 1^{st} expression (on the right of eq.(32)) $A_1^{\alpha_1 term_1} = 1^{-x_0}$,$A_2^{\alpha_2 term_2} = 1^{-iy_0}$ and in the 2^{nd} :$A_1^{\alpha_1 term_1} = (2m)^{-x_0}$ whereas $A_2^{\alpha_2 term_2} = (2m)^{-iy_0}$ so that for the right hand side of eq (31) we arrive at:

$L_1(1^{-x_0})$ +$L_2(1^{-iy_0})$ + $L_1[(2m)^{-x_0}]$ + $L_2[(2m)^{-iy_0}]$=$L_1(1^{-x_0})$ +$L_1[(2m)^{-x_0}]$

$$ =L_1[1\ (1^{-x_0})]\ \ +L_1[1\ ((2m)^{-x_0})] \qquad (32) $$

where the terms in L_2 vanish and A_{11} =1 in both brackets in eq(32) while A_{12} takes on the values 1 and 2m respectively .

Next we decompose L_1 as in equations (6a) , (7a) and (7b);

$$ L_1[1(1^{-x_0})] + L_1[1((2m)^{-x_0})] = \{ L_{11}[1]\}\{ L_{12}[1^{-x_0}]\} + \{L_{11}\ [1]\}\{L_{12}[(2m)^{-x_0}]\} $$

$$ = -2\mathrm{m}x_0 \quad . \qquad (33) $$

The right hand side of equation (30) becomes:

$$ -{}_{(2m)}^{(2)}L[\frac{1}{(4m)^{x_0+iy_0}}] = -{}_{(2m)}^{(2)}L[(4m)^{-x_0}(4m)^{-iy_0}] = -L_1(4m)^{-x_0}\ -L_2(4m)^{-iy_0} $$

$$ = -L_1(2^{-x_0}(2m)^{-x_0}) = -\{L_{11}(2)^{-x_0}\}\{L_{12}(2m)^{-x_0}\} = -(-2x_0)(2\mathrm{m})(-x_0)\delta_{2m,2m} $$

$$= -4\mathrm{m}x_0^2 \quad . \qquad (34)$$

Setting eq.(33) equal to eq.(34) we see that:

$$x_0^2 - \frac{1}{2}x_0 = 0\ . \qquad (35)$$

Clearly $x_0 = \frac{1}{2}$. The trivial solutions x_0 = 0 and $x_0 = \infty$ also are solutions to equation (35).

In case b): n = n´ ≠ 2m

giving rise to the following equation:

$${}^{(2)}_{(2m)}L\left[\frac{1}{1^{x_0+iy_0}} + \frac{1}{(2m)^{x_0+iy_0}}\right] = -\ {}^{(2)}_{(2m)}L\left[\frac{1}{(2n\prime)^{x_0+iy_0}}\right] \ . \quad (36)$$

Equation (36) yields -2mx_0=0 , which is redundant, thus excluding n= n´≠2m as a meaningful choice.

Hence the only choice that bears fruit is case a): n=2m. Therefore

except in trivial cases $x_0 = \frac{1}{2}$.

6. Discussion and Conclusion

We have developed the Dirac equation, from Einstein's Extreme Relativistic Equation along with heuristic quantization, and proven that except in trivial cases x_0 = ½ where x_0 is the real part of the zero of the Riemann Zeta Function.

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